Enhanced Dynkin diagrams and Weyl orbits

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Abstract

The root system Σ of a complex semisimple Lie algebra is uniquely determined by its basis (also called a simple root system). It is natural to ask whether all homomorphisms of root systems come from homomorphisms of their bases. Since the Dynkin diagram of Σ is, in general, not large enough to contain the diagrams of all subsystems of Σ, the answer to this question is negative. In this paper we introduce a canonical enlargement of a basis (called an enhanced basis) for which the stated question has a positive answer. We use the name an enhanced Dynkin diagram for a diagram representing an enhanced basis. These diagrams in combination with other new tools (mosets, core groups) allow to obtain a transparent picture of the natural partial order between Weyl orbits of subsystems in Σ. In this paper *we consider only ADE root systems* (that is, systems represented by simply laced Dynkin diagrams). The general case will be the subject of the next publication.

Contents

1   Introduction

A root system Σ of a complex semisimple Lie algebra $\mathfrak{g}$ is a finite subset of a linear space $\mathfrak{h}(\mathbb{R})^*$ dual to the real form $\mathfrak{h}(\mathbb{R})$ of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The Killing form allows to identify $\mathfrak{h}(\mathbb{R})^*$ with $\mathfrak{h}(\mathbb{R})$ and to interpret it as a Euclidean space. We say that $\widetilde{\Sigma} \subset \Sigma$ is a subsystem of Σ if it is a root system in its span and, for every $\alpha, \beta \in \widetilde{\Sigma}$, $\alpha + \beta \in \Sigma$ implies $\alpha + \beta \in \widetilde{\Sigma}$. To every such $\widetilde{\Sigma}$ there corresponds a semisimple subalgebra $\widetilde{\mathfrak{g}} \subset \mathfrak{g}$ generated by root vectors $e_\alpha \in \widetilde{\mathfrak{g}}$ where $\alpha \in \widetilde{\Sigma}$. Subalgebras $\widetilde{\mathfrak{g}}$, called *regular*, play a fundamental role in the classification of all semisimple subalgebras of semisimple Lie algebras [Dyn52b]. There is a natural partial order between conjugacy classes of regular subalgebras of $\mathfrak{g}$. Namely, $C_1 \prec C_2$ if $\mathfrak{g}_1 \subset \mathfrak{g}_2$ for some $\mathfrak{g}_1 \in C_1$ and $\mathfrak{g}_2 \in C_2$. Enhanced Dynkin diagrams appeared as a result of our attempts to understand better this order.

A classification of complex semisimple Lie algebras is provided by the list of Dynkin diagrams. The Dynkin diagram Γ of $\mathfrak{g}$ has nodes representing roots in a basis $\Pi \subset \Sigma$ and bonds describing relations between the roots. Subdiagrams of Γ are Dynkin diagrams of regular subalgebras of $\mathfrak{g}$. However, not all regular subalgebras can be obtained this way. Besides, non-conjugate regular subalgebras can have identical Dynkin diagrams. Both problems can be efficiently solved by using enhanced Dynkin diagrams.

In this paper we associate a diagram with every subset Λ of Σ. If Λ is a basis, it coincides with the Dynkin diagram Γ of $\mathfrak{g}$. We construct an enhancement Δ of Γ by a recursive procedure which we call the completion. At each step, an extra node is introduced and connected by bonds with a proper part of already introduced nodes. Enhanced Dynkin diagrams for simple ADE algebras $\mathfrak{g}$ are presented on Figure 2.[1]

Conjugacy classes of regular subalgebras of $\mathfrak{g}$ (with respect to the group of inner automorphisms of $\mathfrak{g}$) correspond bijectively to conjugacy classes of subsystems of Σ (with respect to the Weyl group W of Σ). The correspondence preserves the natural

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[1]Notation of roots is coordinated with that on Dynkin and extended Dynkin diagrams shown on Figure 1.

partial order. Group W acts on the set $\mathbb{P}$ of bases of all subsystems $\widetilde{\Sigma} \subset \Sigma$. Following [Dyn52b], we use a name Π-*systems* for elements of $\mathbb{P}$. Every Π-system is contained in a unique subsystem of Σ. This implies a bijection between W-orbits in $\mathbb{P}$ and classes of conjugate subsystems of Σ.

A crucial step is a transaction from elements α of Σ to pairs $(\alpha, -\alpha)$. We call them *projective roots*. To every root α there corresponds a projective root $(\alpha, -\alpha)$, and to every set $\Lambda \subset \Sigma$ there corresponds a set of projective roots $p(\Lambda)$. We put $p(\Sigma) = S$ and we call $p(\Lambda) \subset S$ a projective basis if Λ is a basis of Σ. Projective subsystems and projective Π-systems of S are defined in a similar way. Map p induces a bijection between the class of all subsystems of Σ and the class of all projective subsystems. It also induces a bijection between W-orbits in $p(\mathbb{P})$ and classes of conjugate subsystems of Σ.

If $L \subset S$ and if $|L| = k$, [2] then $L = p(\Lambda)$ for 2^k sets Λ obtained from L by selecting one element from every pair $(\alpha, -\alpha)$. All Λ have the same diagram which we assign as the diagram of L. We call a set of p rojective roots an *enhanced basis* if its diagram is isomorphic to the enhanced Dynkin diagram of Σ. All enhanced bases are conjugate. Besides, every projective Π-system is conjugate to a subset of an enhanced basis. We reduce the classification of all subsystems to the classification of orthogonal subsystems, i.e. those with bondless Dynkin diagrams. One of our tools is a Reduction Lemma which specifies in any projective Π-system L_1 elements a with the property: if f is an isometry of L_1 onto $L_2 \subset S$ and if f coincides with a $\widetilde{w} \in W$, on $L_1 \setminus \{a\}$, then $f = w$ on L_1 for some $w \in W$. [3]

A special role belongs to maximal orthogonal subsets M of Σ, which we call them *mosets*. All mosets in Σ are conjugate. To classify orthogonal subsystems in a root system Σ, we consider a subgroup of W formed by $w \in W$ stabilizing M (we call it a *core group*). We describe it in terms of the enhanced Dynkin diagram of Σ and we apply this result to characterize conjugacy classes of subsystems $\widetilde{\Sigma} \subset \Sigma$ (and even homomorphisms $\widetilde{\Sigma} \to \Sigma$) as well as a partial order between these classes. A byproduct of this investigation is an alternative way to obtain a known classification of regular subalgebras.

1.1 Organization of the paper

We deal with several classes of objects: $\mathbb{A}$ – subsets of a root system Σ; $\mathbb{B}$ – subsets of a projective root system S; $\mathbb{C}$ – diagrams of $\mathbb{A}$ and $\mathbb{B}$. The Weyl group W acts on $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$. Our goal is to investigate W-orbits in $\mathbb{A}$. Classes $\mathbb{B}$ and $\mathbb{C}$ are tools to this end.

Root subsystems, bases (simple root systems), extended bases, Π-systems are subclasses of $\mathbb{A}$ commonly used in the theory of Lie algebras. All of them have shadows in $\mathbb{B}$. If this can cause no confusion, we apply the same name and the same notation for each subclass and for its shadow. For instance, notation $\mathbb{P}$ is used for the

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[2] We denote by $|L|$ the cardinality of a set L.

[3] In general, $w \neq \widetilde{w}$.

class of Π-systems in $\mathbb{A}$ and in $\mathbb{B}$, [4] and a name D_4-sets is used for subsets of Σ and for subsets of S with the Dynkin diagram of type D_4. We keep the same convention for concepts like mosets and core groups introduced in this paper.

Often a statement formulated in the language of one of classes $\mathbb{A}, \mathbb{B}, \mathbb{C}$ can be easily translated into the language of another class. We may state a proposition in a language of $\mathbb{A}$, prove it in terms of $\mathbb{B}$ and apply it in the setting of $\mathbb{C}$. This is done, for instance, in Section 4 where Classification Theorems 4.1, 4.2, 4.3 are stated in terms of root systems, proved in terms of projective root systems and applied to description of Weyl orbits in terms of enhanced Dynkin diagrams. In Section 2 completion is defined for subsets of Σ but constructed by using subdiagrams of an enhanced Dynkin diagram.

In Section 1 we place the basic diagrams after Historical notes because references to them are spread over the entire paper.

We start Section 2 with introducing diagrams Γ of $\Lambda \subset \Sigma$ and Δ of $L \subset S$. We define a class of complete subsets of Σ and we introduce a completion $\overline{X}$ of $X \subset \Sigma$ as a minimal complete extension of X. If X is a basis, then $\overline{X}$ is an enhanced basis and its diagram is an enhanced Dynkin diagram of Σ. We denote it $\Delta(\Sigma)$. A recursive procedure for constructing a completion is illustrated by an example on Figure 3.We conclude the section by proving important properties of enhanced and extended bases and automorphisms of their diagrams. In particular, we establish that every Π-system is conjugate to a subset of an enhanced basis.

A crucial role in investigating which Π-subsystems of an enhanced basis are conjugate is played by Reduction Lemma that is proved at the beginning of Section 3. The rest of this section is devoted to a description of mosets and core groups for all irreducible root systems.

In the Appendix we formulate basic facts on root systems, their bases and automorphisms and on the Weyl groups. For proofs we refer to the literature. At the end of the Appendix we list notation used in the paper.

1.2 Historical notes

An initial motivation for investigating subalgebras of Lie algebras was the problem of a classification of primitive transformation groups posed by S. Lie and solved by him in spaces of 1, 2 and 3 dimensions. The case when the group is not semisimple was examined by V. V. Morozov in 1939.

Investigating semisimple subalgebras in arbitrary Lie algebras can be reduced to studying such subalgebras in semisimple algebras. For classical algebras A_n, B_n, C_n, D_n semisimple subalgebras were studied by Malcev [Mal44]. The case of exceptional algebras G_2, F_4, E_6, E_7, E_8 was treated by Dynkin [Dyn52b]. A principal step in this treatment was the classification of all regular semisimple subalgebras. For every simple Lie algebra, a list of elements $\Lambda_1, \ldots, \Lambda_m$ of $\mathbb{P}$ was given such that every Weyl orbit in $\mathbb{P}$ is represented by subsets of $\Lambda_k, k = 1, \ldots, m$. It was determined by direct case by case computations which of these subsets represent the same orbit. Only

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[4]However, to avoid confusion, we write sometimes $\mathbb{P}(\Sigma)$ and $\mathbb{P}(S)$.

the results of these computations are presented in [Dyn52b]. The results of [Dyn52b] were used by a number of authors.

Recently Oshima [Osh07] verified them in his own way by introducing as a new tool the relation between orbits of Λ and of its orthogonal complement in Σ. His paper contains a number of useful tables.

Dynkin in [Dyn52b] applied his method also to subalgebras of classical algebras. However some time ago he discovered that his description of orbits in D_{2m} (more explicit than that in [Mal44]) is incomplete. He also wanted to have a picture of the partial order between subalgebras which is very difficult to get from the classification in [Dyn52b]. A discussion of these problems resulted in the beginning of the collaboration between the authors of the present article. The ideas of introducing projective roots, enhanced Dynkin diagrams and mosets should be credited to the second author.

Remarkably, slight modifications of enhanced Dynkin diagrams appeared recently in a paper of McKee and Smyth [MS07]. Their diagrams served for a description of all integer symmetric matrices with all eigenvalues in the interval $[-2, 2]$. Investigation of the relation between the two classes of diagrams may be a subject of the future work.
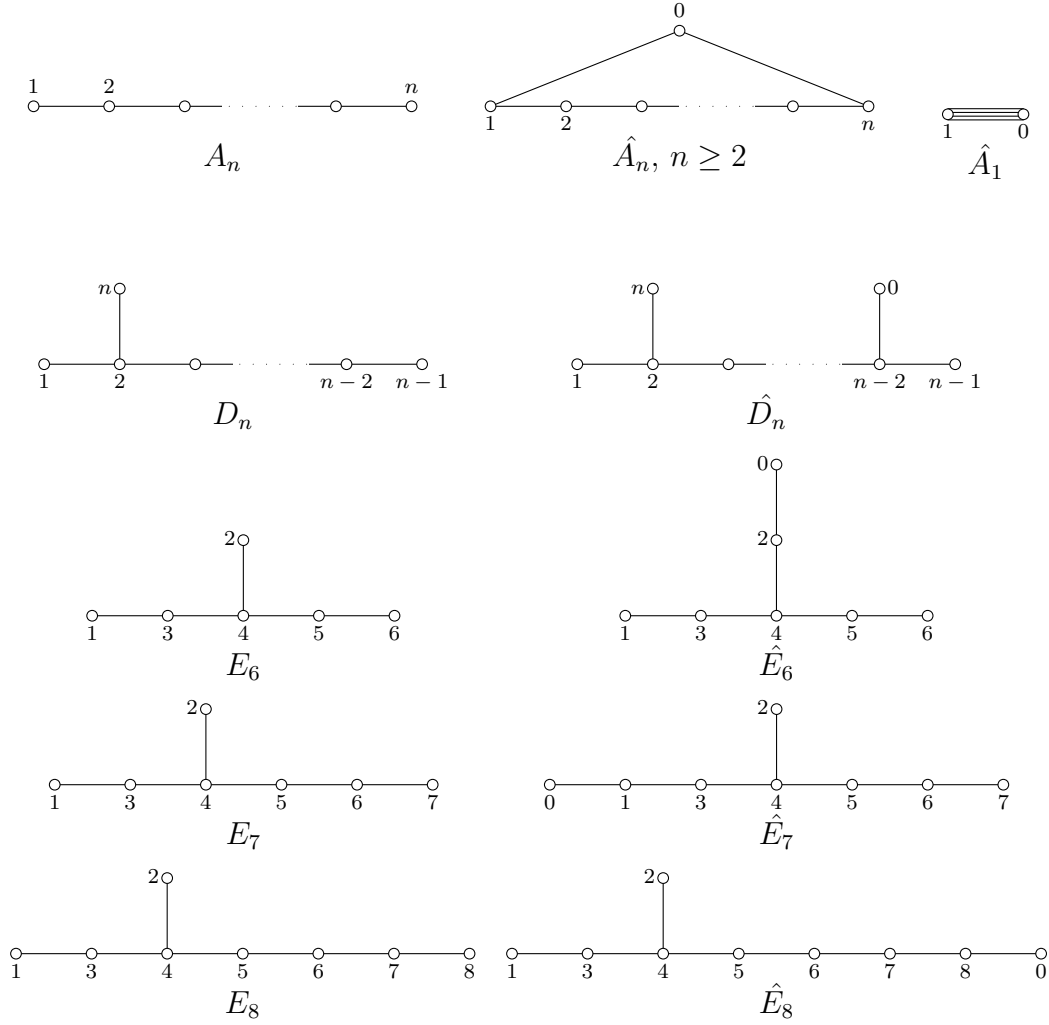
1.3   Basic diagrams

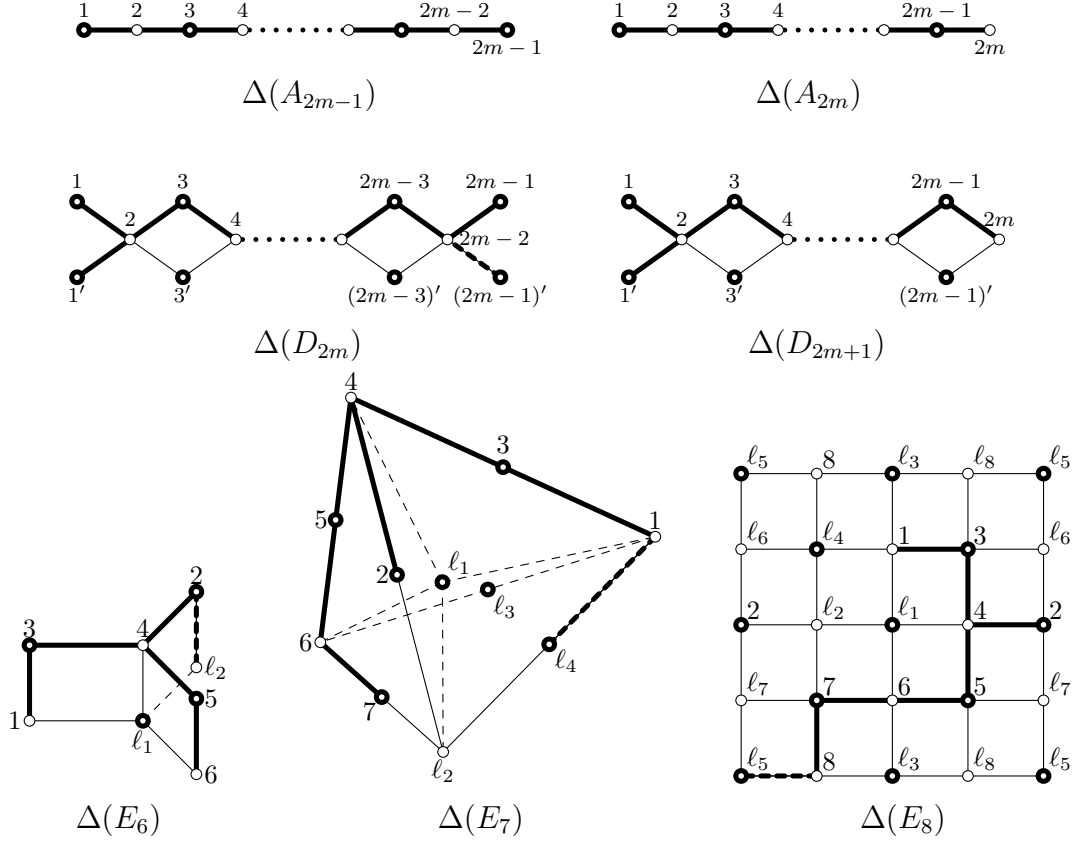Figure 1: Dynkin diagrams and extended Dynkin diagrams of ADE systems

Figure 2: Enhanced diagrams for ADE root systems

For every enhanced Dynkin diagram Δ one of Dynkin diagram $\Gamma \subset \Delta$ is marked by boldfacing its bonds. Except D_n, we use for nodes of Γ the same labels as on Figure 1. The extra nodes for E_6, E_7 and E_8 are labeled $\ell_1, \ell_2, \ldots$ in the order they are introduced by an algorithm described in Subsection 2.2. We draw separate diagrams for D_{2m+1} and D_{2m}. Label $1'$ is substituted for n and extra nodes are labeled $3', 5', \ldots, (2m-1)'$. For E_6 and E_7 it is convenient to embed the enhanced diagrams into the 3-dimensional space. In the case of E_7 the nodes are represented by the vertices of a tetrahedron, by the centers of its edges and by the center of the tetrahedron. Dashed lines are used for bonds shaded by faces. The enhanced diagram for E_8 is a 4×4 lattice on the torus: each bottom node represents the same root as the corresponding top node and a similar identification is made for the left and right sides of the square.

The diagram $\Delta(E_8)$ can be also interpreted as vertices and edges of a hypercube in $\mathbb{R}^4$. $\Delta(E_7)$ can be obtained by eliminating any vertex and all edges containing it. To get $\Delta(E_6)$ one needs to drop any edge and all edges on 2-faces that contain this edge.

Boldfaced nodes represent a maximal orthogonal subsystem of S.

2 Enhanced Dynkin diagrams

Starting from this point we consider only ADE root systems Σ. In this case the Cartan integers $(\alpha|\beta) = \frac{2(\alpha,\beta)}{(\beta,\beta)}$ are equal to 0, 1 or -1 for all non-collinear $\alpha, \beta \in \Sigma$.

2.1 Diagrams and projective diagrams

We define a *diagram* Γ_Λ of $\Lambda \subset \Sigma$ by assigning a node to every $\alpha \in \Lambda$ and by connecting two nodes representing roots $\alpha, \beta \in \Lambda$ by a single bond, if $(\alpha, \beta) \neq 0$ and $\alpha \neq -\beta$, and by a quadruple bond, if $\alpha = -\beta$.[5] We also introduce a *projective diagram* Δ_Λ of Λ which is obtained from Γ_Λ by identifying two nodes corresponding to every pair $\alpha, -\alpha$ (and by dropping the quadruple bond connecting the nodes). Two nodes of Δ_Λ are connected by a bond if and only if they represent non-orthogonal roots. We call such nodes (and roots) *neighbors*. Subdiagrams of any diagram are in a 1-1 correspondence with subsets of nodes.

A subset $\Lambda \subset \Sigma$ is called *symmetric*, if $\Lambda = -\Lambda$. There is a 1-1 correspondence between symmetric subsets of Σ and all subsets of the projective root system S.[The latter correspond bijectively to subdiagrams of the projective diagram of Σ.]

A set of roots is called *irreducible* if it's diagram is connected. Every set $\Lambda \subset \Sigma$ is the union of its maximal irreducible subsets. We call them *irreducible components* of Λ.

A linearly independent subset Λ of Σ is a basis of a subsystem $\widetilde{\Sigma}$ if and only if $\alpha - \beta \notin \Sigma$ for all $\alpha, \beta \in \Lambda$. Recall that we use the name Π-*systems* for such Λ and that we denote the class of all Π-systems by $\mathbb{P}$.

A partial order in Σ is associated with a basis Π by the condition: $\alpha \prec \beta$ if $\beta - \alpha$ can be represented as the sum of elements of Π. If Σ is irreducible, then there is a unique minimal root $\alpha \in \Sigma$ with respect to this order ([Bou02, Chapter VI, Proposition 25(i)]). The set

$$\widehat{\Pi} = \Pi \cup \{\alpha\},$$

is called an *extended basis*. The diagram $\widehat{\Gamma} = \Gamma_{\widehat{\Pi}}$ is the *extended Dynkin diagram* of Σ. Similarly, we define *extended Π-system* $\widehat{\Lambda}$ for every irreducible Π-system Λ.

We say that nodes $a_1, a_2, \ldots, a_k$, $k \geq 3$ of a diagram form a cycle if a_i is a neighbor of a_{i+1} for $1 \leq i \leq k-1$, and a_k is a neighbor of a_1. A diagram is *acyclic*, if it contains no cycles. We say that a root ε is an *end* of $\Lambda \subset \Sigma$ if it has no more than one neighbor. Every root set with an acyclic diagram contains an end.

Proposition 2.1. *Let Φ be a symmetric subset of Σ such that Δ_Φ is connected and acyclic. Then $\Phi = \Lambda \cup -\Lambda$ where Λ is either a Π-system or an extended Π-system.*

Proof. Choose any $\Lambda \subset \Phi$ such that $\Lambda \cap -\Lambda = \emptyset$ and $\Lambda \cup -\Lambda = \Phi$. Since $\Gamma_\Lambda = \Delta_\Phi$ is acyclic, we can assume (by replacing α by $-\alpha$ for some $\alpha \in \Lambda$) that $(\alpha, \beta) \leq 0$ for all $\alpha, \beta \in \Lambda$. Now Proposition 2.1 follows immediately from [Dyn52b, Lemma 5.1]. $\square$

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[5]The number of bonds is equal to the integer $(\alpha|\beta)(\beta|\alpha)$ which can be only 0,1 or 4 in case of ADE root systems.

Proposition 2.1 implies that all connected acyclic diagrams are presented on Figure 1.

2.2 Completion

Definition 2.1. A symmetric subset $\Phi \subset \Sigma$ is called *complete* if the condition $\Lambda \subset \Phi$ implies $\widehat{\Lambda} \subset \Phi$ for every D_4-set $\Lambda \in \mathbb{P}$ of type . The *completion* $\overline{X}$ of $X \subset \Sigma$ is the intersection of all complete subsets of Σ containing X.

Clearly, $\overline{X} \subset \overline{Y}$ if $X \subset Y$.
To construct $\overline{X}$ we introduce a sequence of symmetric sets

$$\Phi_0 \subset \Phi_1 \subset \ldots \Phi_i \cdots \subset \Phi_k$$

where $\Phi_0 = X$, $\Phi_k = \overline{X}$ and, for every incomplete Φ_{i-1}, a diagram Δ_{Φ_i} is defined as an elementary extension of $\Delta_{\Phi_{i-1}}$.

An elementary extension Δ' of any diagram Δ is obtained by extending a D_4-subdiagram $D \subset \Delta$ to a $\widehat{D}_4$-subdiagram of Δ_Σ. Δ' contains an extra node ℓ connected by a bond with the branching node of D and with every node of $\Delta \setminus D$ that has 1 or 3 neighbors among the ends of D. The procedure is motivated by the following proposition.

Proposition 2.2. *Let Λ be a $\widehat{D}_4$-subset of a symmetric set Φ. If $\beta \in \Sigma \setminus \Lambda$ and if $\Lambda \cup \{\beta\}$ is irreducible, then β has two or four neighbors among end elements of Λ.*

Proof. If $\Lambda = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \gamma\}$ with $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ representing the end nodes, then $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + 2\gamma = 0$. [6] This implies

$$(\alpha_0|\beta) + (\alpha_1|\beta) + (\alpha_2|\beta) + (\alpha_3|\beta) + 2(\gamma|\beta) = 0.$$

Taking this relation modulo 2, we obtain that β is the neighbor of an even number of the ends in Λ. □

On Figure 3 we illustrate this procedure on an example of a symmetric set X described by the Dynkin diagram Δ_0 of E_7. We start with its D_4-subdiagram $\{5, 2, 3, 4\}$ (in notation of Figure 1). We get a diagram Δ_1 by introducing an extra node ℓ_1 connected with nodes 1, 4 and 6. In Δ_1, we choose a D_4-subdiagram $\{1, 4, 6, \ell_1\}$ [7] and we introduce an extra node ℓ_2 connected with ℓ_1, 2 and 7. The result of the second step is Δ_2. We arrive at Δ_3 by extending a subdiagram $\{4, \ell_1, \ell_2, 6\}$ of Δ_2 by a node ℓ_3. Finally, we get a diagram Δ_4 by introducing ℓ_4 connected with ℓ_2 and 1. Δ_4 represents $\overline{X}$. (A different selection of elementary extensions would lead to an isomorphic diagram with a permutation of labels $\ell_1, \ell_2, \ell_3, \ell_4$.)

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[6]To prove this, evaluate the squared length of the left side.

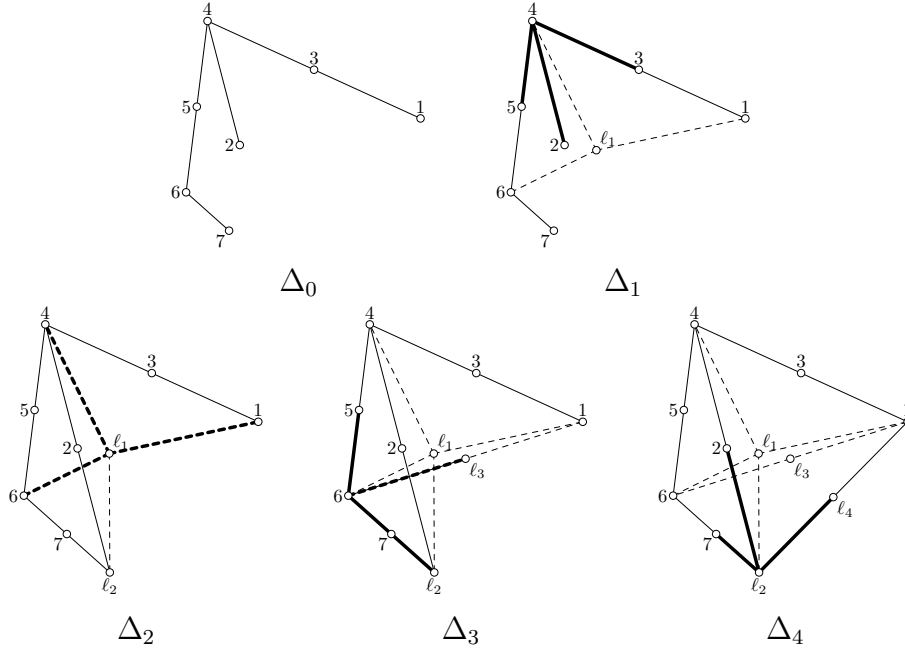[7]Another possible choice is $\{\ell_1, 5, 7, 6\}$.

Figure 3: Detail for E_7

2.3 Enhanced bases and enhanced Dynkin diagrams

Definition 2.2. An *enhanced basis* of Σ is the completion of a basis of Σ. Its projective diagram is called an *enhanced Dynkin diagram* of Σ.

Conjugacy of all bases implies conjugacy of all enhanced bases, hence isomorphism of all enhanced Dynkin diagrams of Σ. These diagrams for irreducible ADE root systems are presented on Figure 2.

A fundamental property of enhanced bases is established in the following theorem.

Theorem 2.1. *Any Π-system is conjugate to a subset of an enhanced basis.*

In the proof of this property we use a class of transformations of $\mathbb{P}$ introduced in [Dyn52b].

Definition 2.3. An *elementary transformation* of an irreducible Π-system Λ is a set $\Lambda^\alpha = \widehat{\Lambda} \setminus \{\alpha\}$, where $\alpha \in \widehat{\Lambda}$. An elementary transformation of an arbitrary Π-system Λ is an elementary transformation of one of its irreducible components. We call this transformation *trivial* if Λ^α is isomorphic to Λ.[8]

It follows from [Dyn52b, Theorems 5.2, 5.3] that:

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[8]A bijection $\varphi : \Lambda_1 \to \Lambda_2$ is an isomorphism if it preserves the bracket $(\cdot|\cdot)$.

Proposition 2.3. *Let Σ be a root system of rank n. Every $\Lambda \in \mathbb{P}$ with $|\Lambda| = n$ can be obtained from any basis of Σ by a sequence of elementary transformations. If Σ is a system of rank n, then any $\Lambda \in \mathbb{P}$ with $|\Lambda| < n$ is contained in a Π-system with n elements.*

Proposition A.6 implies:

Proposition 2.4. *All trivial elementary transformations of $\Lambda \in \mathbb{P}$ are conjugate.*

Proposition 2.5. *Let $\Sigma = D_n$[9] and $2 \leq k \leq n - 2$.[10] Then all Π-systems of type $D_k + D_{n-k}$ in Σ are conjugate.*

Proof. Fix a basis Λ. Note that if Λ^α has type $D_k + D_{n-k}$, then α corresponds to either label k or to $n - k$ on Figure 1. Clearly, these two Π-systems are conjugate by a $w \in W_{\widehat{\Pi}}$ moving node 1 to $n - 1$. [Such w exists by Proposition A.6.] Finally, note that a Π-system of type $D_k + D_{n-k}$ cannot be obtained from Λ by a sequence of more than one non-trivial elementary transformations, since the number of irreducible components of the transformation increases after each step. $\square$

Proof of Theorem 2.1. Let $\Phi \subset \Sigma$ be an enhanced basis. We need to demonstrate that $\mathbb{P}$ is contained in the class $\mathcal{L}$ of Λ with the property: $w(\Lambda) \subset \Phi$ for some $w \in W$. Class $\mathcal{L}$ is preserved under the action of W and it contains, with every Λ, all its subsets. Since Φ is the completion of a basis, we need only to demonstrate that $\mathcal{L}$ contains, with every $\Lambda \in \mathbb{P}$, all elementary transformations of Λ. It is sufficient to check this for irreducible Λ. For $\Lambda = D_{2m}$, E_6, E_7 or E_8, this is true because Φ contains $\widehat{\Lambda}$ (see Figure 2). For type A_n all elementary transformations are trivial, hence they belong to $\mathcal{L}$ by Proposition 2.4. Finally, for type D_{2m+1} all elementary transformations are of type $D_{k_1} + D_{k_2}$ with $k_1 + k_2 = 2m + 1$, and it is clear from Proposition 2.5 and Figure 2 that they belong to $\mathcal{L}$. $\square$

For any group G of transformations of Σ and any $X \subset \Sigma$, we put $G_X = \{g \in G : g(X) = X\}$. In particular, W_X stands for the subgroup of the Weyl group preserving X.

Proposition 2.6. *All bases of Σ contained in an enhanced basis Φ are conjugate by $w \in W_\Phi$.*

Proof. Suppose $\Phi = \overline{\Pi}$, where $\Pi \subset \Sigma$ is a basis. For any other basis $\Pi' \subset \Phi$, by Proposition A.1, there is a $w \in W$ such that $w(\Pi) = \Pi'$ and therefore $w(\overline{\Pi}) = \overline{\Pi'}$. Since a complete Φ contains Π', it contains $\overline{\Pi'} = w(\Phi)$, and $w(\Phi) = \Phi$ because $|w(\Phi)| = |\Phi|$. $\square$

Proposition 2.7. *Let Π be a basis of Σ and $\Phi = \Pi$, $\widehat{\Pi}$ or $\overline{\Pi}$. Every automorphism of Δ_Φ is induced by an automorphism of Σ. The latter can be chosen in W, if Σ is not of type D_{2n}.*

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[9]We abuse notation by writing, e.g. $X = D_n$ if X is a root system, projective root system or a Π-system of type D_n.

[10]We have $D_2 = A_1 + A_1$.

Proof. Let f be an automorphism of $\Delta = \Delta_\Phi$. The Dynkin diagrams $\Gamma = \Gamma_\Pi$ and $\Gamma' = f(\Gamma)$ are subdiagrams of Δ, and $\Gamma' = \Gamma_{\Pi'}$ for a basis $\Pi' \subset \Phi$. By Proposition 2.6, $w(\Pi') = \Pi$ for some $w \in W_\Phi$ and therefore $w(\Gamma') = \Gamma$ and $wf(\Gamma) = \Gamma$. By Proposition A.4, the map wf is induced by an automorphism φ of Σ. Automorphism $\psi = \varphi^{-1}wf$ of Δ is trivial on Γ. This implies cases $\Phi = \Pi$ and $\Phi = \widehat{\Pi}$. If $\Phi = \overline{\Pi}$, then Δ is the enhanced Dynkin diagram of Σ. There exists a sequence of elementary extensions $\Gamma = \Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_k = \Delta$. We prove by induction that ψ is trivial on Δ_i for $i = 0, \ldots, k$. Therefore f is induced by $w^{-1}\varphi \in \operatorname{Aut}\Sigma$.

If $\Sigma = D_{2n}$, then, by Proposition A.5, $\operatorname{Aut}\Sigma$ is generated by W and $-\operatorname{Id}$. Since $-\operatorname{Id}$ acts trivially on Δ, $\operatorname{Aut}\Sigma$ and W induce the same group of transformations on Δ. $\qquad\square$

3   Mosets and core groups

3.1   Reduction Lemma

We consider a projective root system S and we put

$$\langle a|b \rangle = |(\alpha|\beta)|,$$

for every $a = (\alpha, -\alpha), b = (\beta, -\beta) \in S$. By replacing $(\alpha|\beta)$ by $\langle a|b \rangle$ we introduce the concepts of orthogonality, irreducibility and isomorphism for subsets of S.

We denote $I(\Lambda, \Sigma)$ the set of all embeddings of $\Lambda \subset \Sigma$ into Σ preserving $(\cdot|\cdot)$.[11] For a $f \in I(\Lambda, \Sigma)$ we write $f \in W(\Lambda, \Sigma)$ if there exists a $w \in W$ such that $f = w$ on Λ. Clearly, $W(\Lambda, \Sigma) \subset I(\Lambda, \Sigma)$. In a similar way we introduce classes $W(L, S) \subset I(L, S)$.

Lemma 3.1 (Reduction Lemma). *Suppose $f \in I(L, S)$ and a is the neighbor of an end e of L. Then $f \in W(L, S)$ if $\widetilde{f} \in W(\widetilde{L}, S)$, where $\widetilde{f}$ is the restriction of f to $\widetilde{L} = L \setminus \{a\}$.* [12]

Note that reflections

$$s_\alpha(\beta) = \beta - (\beta|\alpha)\alpha, \quad \beta \in \Sigma,$$

satisfy the condition

$$s_\alpha s_\beta s_\alpha = s_\gamma \quad \text{where } \gamma = s_\alpha(\beta), \quad \text{for every } \alpha, \beta \in \Sigma.$$

Lemma 3.2. *Let $\widetilde{L} \subset L \subset S$, $f \in I(L, S)$, $f(a) = a$ for all $a \in \widetilde{L}$. If $a \in L$ and $\langle a|f(a) \rangle \neq 0$, then the restriction $\widetilde{f}$ of f to $\widetilde{L} \cup \{a\}$ belongs to $W(\widetilde{L} \cup \{a\}, S)$.*

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[11] We will call elements of $I(\Lambda, \Sigma$ embeddings.

[12] This Lemma is false for S substituted by Σ. For instance, suppose Σ is of type A_2, Λ is its basis and $f = -\operatorname{Id}$. Any $\alpha \in \Lambda$ is the neighbor of an end and the restriction $\widetilde{f}$ of f to $\widetilde{\Lambda} = \Lambda \setminus \{\alpha\}$ belongs to $W(\widetilde{\Lambda}, \Sigma)$ but $f \notin W(\Lambda, \Sigma)$.

Proof. Set $b = f(a)$. If $a = b$, then $\widetilde{f} = \mathrm{Id}$. Suppose $a \neq b$. Then there exist $\alpha, \beta \in \Sigma$ such that $p(\alpha) = a$, $p(\beta) = b$ and $(\alpha, \beta) < 0$. Since $(\alpha|\beta) = (\beta|\alpha) = -1$, we have

$$s_\alpha(\beta) = s_\beta(\alpha) = \alpha + \beta = \gamma \in \Sigma$$

and $s_\gamma = s_\alpha s_\beta s_\alpha$.

Let us show that $s_\gamma(a) = b$ and $s_\gamma(\ell) = \ell$ for all $\ell \in \widetilde{L}$. We have

$$s_\gamma(\alpha) = s_\alpha s_\beta s_\alpha(\alpha) = -s_\alpha s_\beta(\alpha) = -s_\alpha s_\alpha(\beta) = -\beta.$$

Hence $s_\gamma(a) = b$. Let $\ell \in \widetilde{L}$. There exists $\lambda \in \Sigma$ such that $p(\lambda) = \ell$ and $(\alpha, \lambda) \leq 0$. Since $|(\alpha|\lambda)| = |(\beta|\lambda)|$ we get $(\gamma|\lambda) = (\alpha|\lambda) + (\beta|\lambda) = 0 \,\mathrm{or} -2$. If $(\gamma|\lambda) = 0$, then $s_\gamma(\lambda) = \lambda$. If $(\gamma|\lambda) = -2$, then $\gamma = -\lambda$, $s_\gamma(\lambda) = -\lambda$ and thus $s_\gamma(\ell) = \ell$. $\square$

Proof of Lemma 3.1. We may assume that $\widetilde{f} = \mathrm{Id}$. Suppose $\widetilde{f} \in W(\widetilde{L}, S)$. If $\langle a|f(a)\rangle \neq 0$, then $f \in W(L, S)$ by Lemma 3.2.

Now suppose $\langle a|f(a)\rangle = 0$. Let $\varepsilon \in \Sigma$ be such that $p(\varepsilon) = e$. Set $f' = s_\varepsilon f \in I(L, S)$ and let $\ell \in \widetilde{L} \setminus \{e\}$. We have $f'(\ell) = \ell$ because $\langle e|\ell\rangle = 0$. Since $\langle a|f'(a)\rangle \neq 0$, $f' \in W(\widetilde{L}, S)$ by Lemma 3.2. Hence $f = s_\varepsilon f' \in W(\widetilde{L}, S)$. $\square$

3.2 Mosets

Definition 3.1. An orthogonal set $M \subset X \subset \Sigma$ is a *moset* in X if none element of $X \setminus M$ is orthogonal to M. A moset O of an irreducible X is *perfect* if $X \setminus O$ is orthogonal and $|O| \geq |X \setminus O|$. A perfect moset of an arbitrary X is the union of perfect mosets of irreducible components of X.

Every Π-system has a perfect moset. Indeed, every basis Π of an irreducible root system is the union of two disjoint orthogonal subsets and at least one of them is a perfect moset in Π.

Mosets and perfect mosets in S are defined similarly. Clearly, $p(M)$ is a moset in S if M is a moset in Σ.

Theorem 3.1. *Let O be a perfect moset in $L \in \mathbb{P}(S)$ and let $f \in I(L, S)$. Then $f \in W(L, S)$ if $\widetilde{f} = f|_O \in W(O, S)$.*

Proof. Dynkin diagrams on Figure 1 demonstrate that any perfect moset in L contains at least one of the end nodes e. Hence it does not contain the neighbor a of e, and we can delete a. As the result we get another perfect moset of a Π-system. Therefore we get Theorem 3.1 by using an induction in $|L \setminus O|$ and by applying Lemma 3.1. $\square$

3.2.A. $M \subset \Sigma$ is a moset if and only if its intersection with every irreducible component of Σ is a moset in this component.

The set $\Psi_X(\Sigma) \subset \Sigma$ of all roots orthogonal to $X \subset \Sigma$ forms a subsystem in Σ. For every $\Lambda \supset X$, we denote by $\Psi_X(\Lambda)$ the set of all $\beta \in \Lambda$ orthogonal to X.

3.2.B. If $\alpha \in M \subset \Sigma$, then M is a moset of Σ if and only if $M \setminus \{\alpha\}$ is a moset of $\Psi_\alpha(\Sigma)$.

Proposition 3.1. *All mosets in Σ are conjugate and therefore they have the same cardinality $\mu(\Sigma)$.*

This follows by induction in rank of Σ from Proposition A.2 and 3.2.A, 3.2.B.

Since every orthogonal set is contained in a moset, we have

3.2.C. An orthogonal set with m elements is a moset if and only if $m = \mu(\Sigma)$.

It follows from 3.2.B that

3.2.D. $\mu(\Sigma) = \mu(\Psi_\alpha(\Sigma)) + 1$.

3.2.E. Values of $\mu(\Sigma)$ for irreducible Σ are given by the table:

Table 1: Cardinality of mosets

$\Sigma:$	A_n	D_n	E_6	E_7	E_8
$\mu(\Sigma):$	q_{n+1}	$2q_n$	4	7	8

where q_n is a maximal integer not exceeding $\frac{n}{2}$.

This is an implication of 3.2.D and the following result.

3.2.F. Types of $\Psi_\alpha(\Sigma)$ for irreducible Σ are given in Table 2.

Table 2: Types of the orthogonal complements to roots

$\Sigma:$	A_n	D_n	E_6	E_7	E_8
$\Psi_\alpha(\Sigma):$	A_{n-2}	$D_{n-2} + A_1$	A_5	D_6	E_7

[Since all roots are conjugate, their orthogonal complements have the same type.]

Proof. Let α be the minimal root with respect to a basis Π of Σ. For every $0 < \beta \in \Sigma$ we have
$$\beta = \sum_{\gamma \in \Pi} k_\gamma \gamma,$$
where $k_\gamma \geq 0$ for all $\gamma \in \Pi$. Therefore $\beta \in \Psi_\alpha(\Sigma)$ if and only if $\sum_{\gamma \in \Pi} k_\gamma (\alpha, \gamma) = 0$. Since α is minimal, $(\alpha, \gamma) \leq 0$ for all $\gamma \in \Pi$. Hence $\beta \in \Psi_\alpha(\Sigma)$ if and only if $k_\gamma = 0$ for all γ such that $(\alpha, \gamma) \neq 0$. Thus, the Dynkin diagram of $\Psi_\alpha(\Sigma)$ is obtained from the extended Dynkin diagram of Σ by eliminating nodes representing the minimal root and all its neighbors. Table 3.2.F follows from Figure 1. $\square$

We deduce from 3.2.F by induction in $|O|$ that:

3.2.G. If O is an orthogonal subset of an irreducible root system Σ, then $\Psi_O(\Sigma)$ contains at most one irreducible component not of type A_1.

We denote this irreducible component by $\Theta_O(\Sigma)$ and we put $\Theta_O(\Sigma) = \emptyset$ if it does not exist.

Importance of 3.2.G for the classification of subsystems is illustrated by the following example.

Example 3.1. We claim that $I(L, S) = W(L, S)$ for every irreducible projective root system S and every its Π-system $L = A_{2m}$. Let e be an end of L and let $f \in I(L, S)$. Since W is transitive on S, we may assume $f(e) = e$. The set $\Psi_e(L)$ is a Π-system of type A_{2m-2} in the projective root system $\Psi_e(S)$. Hence $\Psi_e(L) \subset \Theta_e(S)$. We deduce that $f \in W(L, S)$ by induction in m and Reduction Lemma.

3.3 Core groups

Definition 3.2. Let M be a moset of Σ. Group W_M is called a *core group* of Σ.

It follows from Proposition 3.1 that core groups corresponding to all mosets M are isomorphic. Therefore the type of W_M is determined by the type of Σ and we denote it $\mathrm{Cor}(\Sigma)$. We use notation $\nu(\Sigma)$ for the order of this group.

A core group W_M can be identified with a subgroup of the permutation group $\mathbb{S}(M)$. Indeed, there is a natural homomorphism of W_M to $\mathbb{S}(M)$. It follows from Proposition A.3 that $w = \mathrm{Id}$ if $w \in W$ acts trivially on M because none of roots is orthogonal to M.

3.3.A. Let $\Sigma = \cup_k \Sigma_k$ be a partition of Σ into irreducible components. Then $M_k = M \cap \Sigma_k$ is a moset in Σ_k, and W_M is the direct product of W_{M_k}.

Proposition 3.2. *If $\Lambda \subset M$ and $w(\Lambda) \subset M$ for some $w \in W$, then there exists a $w' \in W_M$ such that $w'(\alpha) = w(\alpha)$ for all $\alpha \in \Lambda$.*

Proof. The set $w(\Lambda)$ belongs to mosets M and $w(M)$. If follows from 3.2.B that $M_1 = M \setminus w(\Lambda)$ and $M_2 = w(M) \setminus w(\Lambda)$ are mosets of the orthogonal complement $\widetilde{\Sigma}$ to $w(\Lambda)$ in Σ. By Proposition 3.1, there is a $\widetilde{w} \in W(\widetilde{\Sigma}) \subset W$ such that $\widetilde{w}(M_2) = M_1$. Since $\widetilde{w}$ acts trivially on $w(\Lambda)$, $w' = \widetilde{w}w$ preserves M. $\square$

3.3.B. Suppose L is a Π-system in S, M is a perfect moset in L and $f \in I(L, S)$. Let O be a moset in L such that $f(O) \subset M$. Then $f \in W(L, S)$ if $f|_M \in W_M(O, M)$.

3.3.C. If Σ is irreducible, then W_M acts transitively on M.

This follows from Proposition A.2 and from Proposition 3.2 applied to singletons.

3.3.D. The isotropy group $(W_M)_\alpha$, where $\alpha \in M$, is isomorphic to the core group of $\Psi_\alpha(\Sigma)$.

This follows from Proposition A.3.

We conclude from 3.3.C and 3.3.D, that:

3.3.E. $\nu(\Sigma) = \nu(\Psi_\alpha(\Sigma))\mu(\Sigma)$.

3.3.F. We have:

Table 3: Orders of core groups

$\Sigma :$	A_n	D_{2m+1}	D_{2m}	E_6	E_7	E_8
$\nu(\Sigma) :$	$\mu!$	$2^m \cdot m!$	$2^{m-1} \cdot m!$	$\mu!$	168	1344

where $\mu = \mu(\Sigma)$.

This follows from 3.3.E and Tables 2, 1.

Suppose Σ is irreducible. Since $W_M \subset \mathbb{S}(M)$, 3.3.F implies:

Theorem 3.2. $W_M = \mathbb{S}(M)$ *if and only if* $\Sigma = A_n$ *or* E_6.

Core groups of D_n, E_7 and E_8 will be described in terms of enhanced Dynkin diagrams. For D_{2m} and D_{2m+1} we represent M by a matrix

$$\begin{pmatrix} 1 & 3 & \dots & 2m-1 \\ 1' & 3' & \dots & (2m-1)' \end{pmatrix}. \tag{3.1}$$

For E_7 and E_8 a new labeling of M is introduced on Figure 4 where the labels are vectors of the space $\mathbb{F}_2^3$ over the field $\mathbb{F}_2$ with elements $0, 1$.
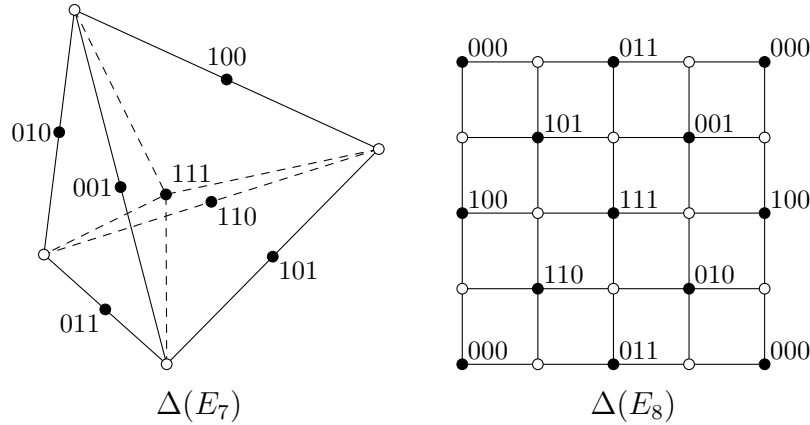

Figure 4: $\Delta(E_7)$ and $\Delta(E_8)$

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Theorem 3.3. *The core group* $\mathrm{Cor}(D_{2m+1})$ *can be represented as the group of permutations of columns in the matrix* (3.1) *combined with transpositions of the entries in some of columns. Permutations with an even number of such transpositions form a subgroup representing* $\mathrm{Cor}(D_{2m})$.

Proof. Transpositions τ_i of $2i - 1$ and $(2i - 1)'$ leaving fixed the rest of the entries generate a subgroup T of $\mathbb{S}(M)$ of order 2^m. Transpositions σ_i of columns containing $2i - 1$ and $2i + 1$ generate a subgroup S of $\mathbb{S}(M)$ of order $m!$. The pair S, T generates a subgroup of order $2^m \cdot m!$ that is equal to W_M by 3.3.F. To prove Theorem 3.3 for D_{2m+1} it is sufficient to demonstrate that τ_i and σ_i belong to W_M. Similarly, to prove Theorem 3.3 for D_{2m+1} it is sufficient to demonstrate that $\tau_i \tau_{i+1}$ and σ_i belong to W_M.

First, we consider the case of D_4. In this case every permutation of M is induced by an automorphism of $\Delta(D_4)$ which, by Proposition 2.7, is induced by an automorphism of Σ. Hence W_M is a normal subgroup of $\mathbb{S}(M)$. There is only one such subgroup of four elements, and it is generated by σ_1, σ_2 and $\tau_1\tau_2$. The case of any $D_{2m}, m \geq 2$, follows from D_4.

In order to prove the Theorem for D_{2m+1}, it suffices to show $\tau_1 \in W_M$. But τ_1 is induced by an automorphism of Δ, hence the statement follows from Proposition 2.7. $\qquad\square$

In the case of E_7, only nonzero vectors of $\mathbb{F}_2^3$ are used for labeling M.

Theorem 3.4. $\mathrm{Cor}(E_7)$ *is represented by* $\mathrm{GL}_3(\mathbb{F}_2)$.

More precisely, we identify $\mathrm{Cor}(E_7)$ with a subgroup $\mathbb{S}_{cor}$ of $\mathbb{S}(M)$ and we demonstrate that $\mathbb{S}_{cor}$ coincides with the group $\mathbb{S}_{gl} = \mathrm{GL}_3(\mathbb{F}_2)$ of all invertible linear transformations of $\mathbb{F}_2^3$.

Proof. Note that $|\mathbb{S}_{gl}| = 168$. By 3.3.F, $\mathbb{S}_{cor}$ has the same order. Therefore it is sufficient to prove $|\mathbb{S}| \geq 168$ for the group $\mathbb{S} = \mathbb{S}_{cor} \cap \mathbb{S}_{gl}$.

Every automorphism σ of $\Delta(E_7)$ preserves the vector 111. We see from Figure 4 that $x + y + z = 0$ if and only if the nodes with these labels lie either on the same face of the tetrahedron or on a line through its center. This property is preserved under σ and therefore $\sigma(x) + \sigma(y) + \sigma(z) = 0$. Since $z = x + y$, we have $\sigma(x + y) = \sigma(x) + \sigma(y)$, which implies $\sigma \in \mathbb{S}_{gl}$. By Proposition 2.7, every automorphism σ is induced by an element of the Weyl group which implies that automorphisms $\sigma_1, \ldots, \sigma_{24}$ determined by 24 rotations of the tetrahedron belong to $\mathbb{S}_{cor}$.

Now we consider an automorphism of $\widehat{D_4}$-set $\{2, 3, 4, 5, \ell_1\} \subset \Delta(E_7)$ on Figure 2 which transposes 3 with ℓ_1 and 2 with 5. By Theorem 3.3 it is induced by a $\tau \in \mathbb{S}_{cor}$ not moving elements $7, \ell_3, \ell_4$. We use Figure 4 to check that $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in M$. Therefore $\tau \in \mathbb{S}_{gl}$.

Since $|M| = 7$ is prime, transformations $\tau, \sigma_1, \ldots, \sigma_{24}$ generate a subgroup of $\mathbb{S}$ acting transitively on M. The order of this subgroup is equal at least to $24 \times 7 = 168$. $\qquad\square$

Definition 3.3. The parity $\rho(O)$ of a subset $O \subset M$ is equal to 0 if the sum of all labels of O is $\mathbf{0}$ and equal to 1 if this sum is not $\mathbf{0}$.

Corollary 3.1. *Two subsets of M with the same cardinality k are conjugate if $k \neq 3, 4$. If $k = 3$ or 4, then they are conjugate if and only if they have the same parity.*

It is sufficient to check this for $k = 1, 2, 3$ because $w(M \setminus O) = M \setminus O'$ if $w(O) = O'$. Note that, for $|O| = 3$, $\rho(O) = 1$ if and only if O corresponds to a basis of $\mathbb{F}_2^3$. The rest follows from Theorem 3.4.

Corollary 3.2. *W_O induces on O the group of all permutations if $|O| = 3$ or if $|O| = 4$ and $\rho(O) = 0$. If $|O| = 4$ and $\rho(O) = 1$, then W_O induces on O the group of all permutations preserving an element α.*[13]
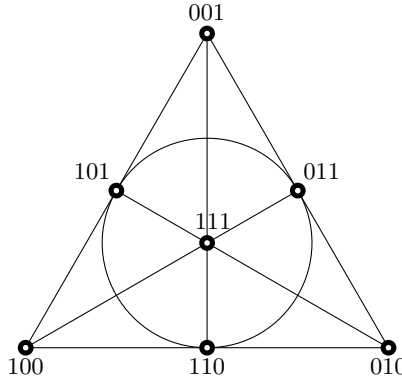

Figure 5: Fano plane

Remark 3.1. By comparing Figure 4 with Figure 5 we see that a moset M can be interpreted as a projective plane over $\mathbb{F}_2$ (Fano plane). Linearly dependent triplets are represented by lines in the plane and $\mathrm{Cor}(E_7)$ is represented by the collinearity group.

Mosets M and $M^\diamond$ of E_7 and E_8 on Figure 4 are related by the formula

$$M^\diamond = M \cup \{\mathbf{0}\}$$

where $\mathbf{0}$ is the zero vector 000.

Theorem 3.5. *$\mathrm{Cor}(E_8)$ is represented by the group $\mathrm{AFF}_3(\mathbb{F}_2)$ of all affine transformations of $\mathbb{F}_2^3$.*

Proof. Denote $\mathbb{S}^\diamond_{aff}$ and $\mathbb{S}^\diamond_{cor}$ subgroups of $\mathbb{S}(M^\diamond)$ representing $\mathrm{AFF}_3(\mathbb{F}_2)$ and $\mathrm{Cor}(E_8)$. Group $\mathbb{S}_{gl} = \mathbb{S}_{cor}$ introduced in proof of Theorem 3.4 is naturally embedded into $\mathbb{S}^\diamond_{aff} \cap \mathbb{S}^\diamond_{cor}$. By 3.3.D and Theorem 3.4, the isotropy group of $\mathbb{S}^\diamond_{cor}$ at $\mathbf{0}$ coincides with $\mathbb{S}_{cor}$. The parallel translation τ by 111 (in $M^\diamond$) is an element of $\mathbb{S}^\diamond_{aff}$. It belongs also to $\mathbb{S}^\diamond_{cor}$ because it is induced by an automorphism φ of $\Delta(E_8)$. [14]

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[13]α is a unique element such that $\rho(O \setminus \{\alpha\}) = 0$.

[14]To define φ we relabel $\Delta(E_8)$ by elements (a, b) of group $\mathbb{F}_4 \times \mathbb{F}_4$ and we put $\varphi(a, b) = (a+2, b+2)$.

Group $\mathbb{S}^\diamond_{aff}$ is generated by $\mathbb{S}_{gl}$ and τ. Group generated by $\mathbb{S}_{cor}$ and τ acts transitively on $M^\diamond$ and therefore its order is at least equal to $8|\mathbb{S}_{cor}| = 8 \cdot 168 = |\mathbb{S}^\diamond_{cor}|$. Since $\mathbb{S}_{gl} = \mathbb{S}_{cor}$, we conclude that $\mathbb{S}^\diamond_{aff} \subset \mathbb{S}^\diamond_{cor}$, and $\mathbb{S}^\diamond_{aff} = \mathbb{S}^\diamond_{cor}$ because both groups have the same order. $\qquad\square$

Corollary 3.3. *Two subsets of M with the same cardinality k are conjugate if $k \neq 4$. If $k = 4$, then they are conjugate if and only if they have the same parity.*

Proof of Corollary 3.3 is similar to that of Corollary 3.1.

A parity $\rho(O)$ of an orthogonal set O is a parity of a $O' \subset M$ conjugate to O. (By Corollaries 3.1 and 3.3, $\rho(O)$ does not depend on the choice O'.)

Corollary 3.4. *If $|O| = 4$, then W_O induces on O the group of all permutations.*

Remark 3.2. Consider a function η defined on irreducible ADE root systems by a table:

$\Sigma:$	A_n	D_n	E_6	E_7	E_8
$\eta(\Sigma):$	$\mu(A_n)$	1	$\mu(E_6)$	2	3

The condition $\eta(\Sigma) < |O| < \mu(\Sigma) - \eta(\Sigma)$ is necessary and sufficient for existence of a set O' isomorphic but not conjugate to an orthogonal set O.

We do not use this fact. It can be proved by using Theorem 3.3, Corollaries 3.1, 3.3 and Table 2. [$\eta(\Sigma)$ is a minimal value of $|O|$ such that $\Psi_O(\Sigma)$ is empty or reducible.]

3.4 Orthogonal subsets of an enhanced basis

Suppose S is a projective root system. We want to classify Weyl orbits in $\mathbb{P}$ by using a classification of orbits of a core group W_M. It is convenient to work not with the action of W on $\mathbb{P}$ but with its action on embeddings of $L \in \mathbb{P}$ into S. Maps $f_1, f_2 \in I(L, S)$ lie on the same Weyl orbit if and only if $f_2 f_1^{-1} \in W(f_1(L), S)$. For a moset $M \subset S$ and a subset $O \subset M$ put $f \in W_M(O, M)$ if $f = w$ on O for a $w \in W_M$. It follows from Proposition 3.2 and Theorem 3.1 that:

Proposition 3.3. *Let $L \in \mathbb{P}$ and $f \in I(L, S)$. For a perfect moset O of L, a moset M of S and any embeddings $f_1 \in W(O, M)$, $f_2 \in W(f(O), M)$ we have*

$$f \in W(L, S) \quad \text{if and only if} \quad f_2 f f_1^{-1} \in W_M(f_1(O), M).$$

Let $\Phi \subset S$ be a projective enhanced basis containing M. We give an algorithm to determine which $f \in I(L, \Phi)$ belong to $W(L, \Phi)$. By Proposition 3.3, it is sufficient for every orthogonal subset $O \subset \Phi$ to construct an embedding $f \in W(O, M)$.

Let $O \subset \Phi$ be an orthogonal subset, $O_2 = O \cap M$ and $O_1 = O \setminus O_2$. The set $\Phi_1 = \Psi_{O_2}(\Phi)$ contains O_1 and $M_1 = M \cap \Phi_1$.

Proposition 3.4. *There is a subsystem $S_1 \subset S$ such that Φ_1 is its enhanced basis and M_1 is its moset.*

Proof. For a singleton O_2 this is clear from Figure 2. For any O_2, we apply induction in $|O_2|$. $\square$

Proposition 3.4 reduces the case of any O to O disjoint from M. It is clear from Figure 2 that $\Phi \setminus M$ is an orthogonal set. Therefore it is sufficient to find f for $O = \Phi \setminus M$. In Table 5 we use labels on Figure 2 to describe $f(a)$ for every $a \in O$.

Table 5: Embeddings f for irreducible S

$S:$	A_{2m} or A_{2m+1}	D_{2m}	D_{2m+1}
$a \in O:$	$2, 4, \ldots, \quad 2m$	$2, 4, \ldots, 2m-2$	$2, 4, \ldots, \quad 2m$
$f(a):$	$1, 3, \ldots, 2m-1$	$1, 3, \ldots, 2m-3$	$1, 3, \ldots, 2m-1$

$S:$	E_6	E_7	E_8
$a \in O:$	$1, 4, 6, \ell_2$	$1, 4, 6, \ell_2$	$1, 4, 6, 8, \ell_2, \ell_6, \ell_7, \ell_8$
$f(a):$	$3, \ell_1, 5, 2$	$3, \ell_1, 5, 2$	$3, \ell_1, 5, \ell_5, 2, \ell_4, 7, \ell_3$

To get this table we use Proposition 2.7. We note that, for A_{2m}, E_6, E_8, f is induced by an automorphism of Φ which is induced by a $w \in W$. For the rest of irreducible S we find an enhanced basis $\widetilde{\Phi} \subset \Phi$ of a subsystem $\widetilde{S}$ of S such that $O \subset \widetilde{\Phi}$ and $\widetilde{\Phi} \setminus O$ is a moset of $\widetilde{S}$. Values of $\widetilde{\Phi}$ are described in the following table:

Table 6: Set $\widetilde{\Phi}$

$S:$	A_{2m+1}	D_{2m}	D_{2m+1}	E_7
$\widetilde{\Phi}:$	$\{1, 2, \ldots, 2m\}$	$\{1, 2, \ldots, 2m-2\}$	$\{1, 2, \ldots, 2m\}$	$\{1, 2, 3, 4, 5, 6, \ell_1, \ell_2\}$

Example 3.2. We use diagram $\Delta(E_8)$ on Figure 2 (representing an enhanced basis Φ) to define a Π-system L of type A_7 and its embedding f into Φ such that f is in $I(L, \Phi)$ but not in $W(L, \Phi)$. Recall that boldfaced nodes represent a moset M for E_8. We put

$$L = \{2, 4, 5, 6, 7, 8, \ell_5\}, \quad f(L) = \{3, 1, \ell_1, \ell_2, 2, \ell_7, \ell_5\}, \quad O = \{2, 5, 7, \ell_5\}.$$

Since O and $f(O)$ have different parity, $f \notin W_M(O, M)$. By Proposition 3.3 applied to $f_1 = f_2 = Id$, $f \notin W(L, \Phi)$.

Example 3.3. Now we define $L \in \mathbb{P}(E_7)$ of type $A_3 + A_1$ and its embedding f into Φ of class $W(L, \Phi)$. We take

$$L = \{7, 6, \ell_3, 4\}, \quad f(L) = \{1, 3, 4, 6\}, \quad O = \{7, \ell_3, 4\}.$$

By applying Proposition 3.4 and Table 5, we find $f_1 \in W(O, M)$ with image $\{7, \ell_3, 5\}$ and $f_2 \in W(f(O), M)$ with image $\{3, \ell_1, 5\}$. Since $f_1(O)$ and $f_2(f(O))$ have the same parity, $f_2 f f_1^{-1} \in W(f_1(O), M)$ by Theorem 3.4. Finally, Proposition 3.3 implies $f \in W(L, \Phi)$.

Remark 3.3. It can be shown that every projective Π-system in S is conjugate to a $L \subset \Phi$ with $O \subset M$.

4 Classification theorems

If X, Y are root sets or projective root sets, then we write $X \simeq Y$ if they are isomorphic and $X \sim Y$ if they are conjugate.

Definition 4.1. A Π-system $\Lambda \subset \Sigma$ is called *normal*, if $\Lambda \simeq \Lambda'$ implies $\Lambda \sim \Lambda'$ for every $\Lambda' \in \mathbb{P}$. Otherwise Λ is called *special*.

Definition 4.2. The *significant part* of a Π-system $\Lambda \subset \Sigma$ is the union of all its irreducible components of types A_{2m-1}. We call Λ *significant* if it coincides with its significant part.

Note that every significant Π-system contains a unique perfect moset. It is insignificant if at least one of its irreducible component is not of type A_{2m-1}.

In this section we prove the following theorems.

Theorem 4.1. *Two Π-systems in an irreducible Σ are conjugate if and only if their significant parts are conjugate.*

Theorem 4.2. *All special Π-systems in an irreducible Σ are significant unless Σ is of type D_n with $n \geq 7$.*

Theorem 4.3. *Suppose an irreducible Σ is not of type $D_n, n \geq 7$. Then two significant Π-systems in Σ are conjugate if and only if their perfect mosets are conjugate.*

To be able to refer to Lemmas 3.1 and 3.2, we prove these theorems restated in terms of projective root systems S and embeddings $f \in I(L, S)$.

4.1 Proof of Theorem 4.1

4.1.A. Suppose an irreducible insignificant $L \in \mathbb{P}$ is not of type D_{2m}. Then $I(L, S) = W(L, S)$.

Proof. For $L = A_{2m}$ this follows from Example 3.1.

If $L \neq A_{2m}$, then it is of type D_{2m+1} with $m > 1$ or E_m with $m = 6, 7, 8$. We use an induction in $|L|$. Let L^e be obtained by eliminating from L an end e. By excluding from consideration one of three ends in each L, [15] we obtain an insignificant $L^e \neq D_{2m}$, and, by induction assumption, $f' = wf|_{L^e} = \mathrm{Id}$ for some $w \in W$. If $\langle f'(e)|e \rangle \neq 0$, then the diagram of $L^e \cup \{e, f'(e)\}$ is acyclic and, by Proposition 2.1, it must be on Figure 1. With our selection of e, this is not the case. Hence $\langle f'(e)|e \rangle \neq 0$ and, by Lemma 3.2, f' and f are in $W(L, S)$. $\square$

4.1.B. If $L \in \mathbb{P}$ is of type D_{2m} and if $f \in I(L, S)$, then $fs \in W(L, S)$ for an automorphism s of L.

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[15] In notation of Figure 1, these are $2m$ for D_{2m+1}, 1 for E_7 and 2 for E_6 and E_8.

Proof. Let e_1 and e_2 be ends of L such that $\widetilde{L} = L \setminus \{e_1, e_2\}$ is of type A_{2m-2}. By 4.1.A, we may assume $f|_{\widetilde{L}} = \mathrm{Id}$. The set $E = \{e_1, e_2, f(e_1), f(e_2)\}$ is not orthogonal. Indeed, otherwise an acyclic diagram of $\widetilde{L} \cup E$ would have 5 ends, which contradicts Proposition 2.1. Without loss of generality, we assume $\langle e_2 | f(e_2) \rangle \neq 0$ or $\langle e_2 | f(e_1) \rangle \neq 0$. In any case, there is a $q \in \mathrm{Aut}\, L$, trivial on $\widetilde{L}$, such that $\langle e_2 | fq(e_2) \rangle \neq 0$. By Lemma 3.2, $wfq|_{L_1} = \mathrm{Id}$ for $L_1 = L \setminus \{e_1\}$ and a $w \in W$.

Set $f' = wfq$. If $\langle f'(e_1) | e_1 \rangle \neq 0$, then $f' \in W(L, S)$ by Lemma 3.2. Hence $fs \in W(L, S)$ for $s = q$. Case $\langle f'(e_1) | e_1 \rangle = 0$ is possible only if $m = 2$. Then $\widehat{L} = L \cup \{f'(e_1)\}$ is the extended projective Π-system and there is a $w' \in W_{\widehat{L}}$ transposing e_1 and $f'(e_1)$ and preserving L_1. Hence $w'f'(L) = L$ and $w'f'$ induces on L an automorphism q_1, and 4.1.B holds for $s = qq_1^{-1}$. $\qquad\square$

Proposition 4.1. *Suppose A is the significant part of $L \in \mathbb{P}$ and $f(A) \sim A$. Then $fs \in W(L, S)$ for an automorphism s of L preserving each irreducible insignificant component and trivial on all of them not of type D_{2m}.*

Proof. We use induction in the number of irreducible insignificant components of L. Let Q be one of them. Put $L' = L \setminus Q$. Suppose $f|_{L'} = \mathrm{Id}$. Let O' be a perfect moset of L'. Then Q and $f(Q)$ are irreducible insignificant Π-systems in irreducible projective root system $S' = \Theta_{O'}(S)$. There is an automorphism q of Q such that $w'f'q = \mathrm{Id}$ where $f' = f|_Q$ and $w' \in W$. Extend q trivially to an automorphism s of L, and set $f'' = w'fs$. If $O \supset O'$ is a perfect moset of L, we have $f''|_O = \mathrm{Id}$. By Theorem 3.1, $f'' \in W(L, S)$, hence $fs \in W(L, S)$. $\qquad\square$

4.2 Proof of Theorem 4.2

We deal with the class $\mathfrak{S}$ of all irreducible projective ADE systems S except $D_n, n \geq 7$. Set $\mathfrak{A}$ to be the class of projective ADE root systems having all special Π-systems significant. In order to prove Theorem 4.2, we have to show $\mathfrak{S} \subset \mathfrak{A}$. We use the following propositions.

Proposition 4.2. *Suppose $L \subset S$ is a significant Π-system with the perfect moset O and $f \in I(L, S)$. Let $L_1 \subset L$ be an irreducible component with the perfect moset O_1 and $L_2 = L \setminus L_1$. The following statements are equivalent:*

(1) there is a $w \in W$ such that $wf(L) = L$ and $wf(L_1) = L_1$;

(2) there is a $w' \in W$ such that $w'f(L_2) = L_2$ and $w'f(O_1) = O_1$.

Proof. (1) $\Rightarrow$ (2) with $w' = w$. Suppose that (2) holds. By substituting $w'f$ for f we can assume that $f(L_2) = L_2$ and $f(O_1) = O_1$. Denote by O_2 the perfect moset of L_2. Note $S_1 = \Psi_{O_2}(S)$ contains L_1 and $f(L_1)$. (Indeed, $f(L_1) \perp f(L_2) = L_2 \supset O_2$.) Since $L_1 = A_k$, by Theorem 3.2, there is a $w_1 \in W(S_1)$ such that $w_1 f = \mathrm{Id}$ on O_1 (and $w_1 = \mathrm{Id}$ on O_2). Let h be an automorphism of L such that $h = \mathrm{Id}$ on L_1 and $fh = \mathrm{Id}$ on L_2. Then we have $w_1 fh = \mathrm{Id}$ on O_1 and O_2, and therefore $w_1 fh = \mathrm{Id}$ on $O = O_1 \cup O_2$. By Theorem 3.1, $w_1 fh = w_2$ on L, where $w_2 \in W$. We put $w = w_2^{-1} w_1$.

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Note that $wfh = \text{Id on} L$ and $w = \text{Id} = hf$ on L_2. Therefore $wf(L) = wfh(L) = L$ and $wf(L_1) = wfh(L_1) = L_1$. $\square$

Proposition 4.3. *If S is irreducible and $\Psi_a(S) \in \mathfrak{A}$ for all $a \in S$, then $S \in \mathfrak{A}$.*

Proof. Assuming that $\Psi_a(S) \in \mathfrak{A}$ for all $a \in S$ we check that, if $L \in \mathbb{P}(S)$ has the significant part $A \neq L$, then $f(L) \sim L$ for all $f \in I(L, S)$. This follows from Theorem 4.1 if $A = \emptyset$.

If A is non-trivial, then we consider its irreducible component A_1 of minimal rank and an end e in A_1. Replacing f by wf, $w \in W$, we can assume $f(e) = e$. Π-system $\widetilde{L} = \Psi_e(L)$ in $\widetilde{S} = \Psi_e(S)$ is insignificant. Hence, by our assumption, $f(\widetilde{L}) = \widetilde{w}(\widetilde{L})$ for a $\widetilde{w} \in W(\widetilde{S})$. The irreducible component $\widetilde{A_1} = \Psi_e(A_1)$ of $\widetilde{L}$ has the smallest rank, and therefore is preserved by $\widetilde{w}$. Then $\widetilde{w}$ preserves $A \setminus A_1$ and the perfect moset of A_1. By Proposition 4.2, $f(L) \sim L$. $\square$

Denote by $\mathfrak{S}_1$ the class $\mathfrak{S}$ augmented by reducible types $A_2 + A_1$, $A_3 + A_1$ and $D_4 + A_1$ and by all orthogonal types $kA_1, k \geq 1$.

4.2.A. If $S \in \mathfrak{S}_1$, then $\Psi_a(S) \in \mathfrak{S}_1$ for all $a \in S$. This follows immediately from Table 2.

4.2.B. Any reducible $S \in \mathfrak{S}_1$ belongs to $\mathfrak{A}$.

Proof. The statement is trivial for orthogonal S. Suppose $S = A_2 + A_1$, $A_3 + A_1$ or $D_4 + A_1$ and let $L \in \mathbb{P}(S)$ be insignificant. If L is irreducible, then L must belong to a unique irreducible component of S not of type A_1, and it is normal by Theorem 4.1. If L is reducible, then $L \supset \widetilde{L}$ where $\widetilde{L}$ is of type $A_2 + A_1$. [A_2 is a subset of an irreducible component of L and A_1 is contained in another component of L.] But none of A_2, A_3, D_4 contains $A_2 + A_1$, hence the singleton in $\widetilde{L}$ coincides with that in S. On the other hand, A_2 is a maximal proper insignificant Π-system in A_3 and D_4. Hence $L = \widetilde{L}$ and the statement follows from normality of A_2. $\square$

Proof of Theorem 4.2. We will prove that $\mathfrak{S}_1 \subset \mathfrak{A}$.[16] We use induction in rank of $S \in \mathfrak{S}_1$. In case of reducible S we have $S \in \mathfrak{A}$ by 4.2.B. If S is irreducible, we use Proposition 4.3 and 4.2.A. $\square$

4.3 Proof of Theorem 4.3

Theorem 4.3 is equivalent to the statement:

Proposition 4.4. *Let $S \in \mathfrak{S}$, $L \in \mathbb{P}(S)$ and $f \in I(L, S)$. If $f(O) = O$ for a perfect moset O of L, then $f(L) \sim L$.*

Proposition 4.4 is trivial if L is normal and it follows from Theorem 3.1 if $W_O = \mathbb{S}(O)$. By Theorem 3.2, it remains to prove it for a special L and S of type D_4, D_5, D_6, E_7, E_8. By Theorem 4.2, L is significant. In the proof we use two lemmas.

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[16]In fact, $\mathfrak{S}_1 = \mathfrak{A}$. We do not use this fact.

Lemma 4.1. *If $L \in \mathbb{P}(S)$ is of type $2A_3$, then there exists a $w \in W$ transposing the irreducible components of L.*

Proof. Only D_6, E_7 and E_8 contain such L. We have $|O| = 4$. Let G_O be the group of transformations of O induced by W_O. It follows from Theorem 3.1 that if $G_O = \mathbb{S}(O) \simeq \mathbb{S}_4$, then $I(L, S) = W(L, S)$. For $S = E_8$, by Corollary 3.4, $G_O \simeq \mathbb{S}_4$. Corollary 3.2 implies that, in the case of E_7, G_O is isomorphic either to $\mathbb{S}_4$ or to $\mathbb{S}_3$. However we deduce from Proposition 2.7 (applied to Φ of type $2A_3$) that G_O contains a subgroup isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$. Hence $G_O \simeq \mathbb{S}_4$.

A unique subdiagram of type $2A_3$ of diagram $\Delta(D_6)$ is $\{1, 2, 1', 5, 4, 5'\}$ (in notation of Figure 2). Hence, $O = \{1, 1', 5, 5'\}$. Let f be an automorphism of L inducing on O transpositions of 1 with 5 and of $1'$ with $5'$. By Theorem 3.3, $f \in W(L, S)$. $\square$

Denote by $\mathfrak{S}_2$ the class of projective root systems $S = S_0 + S_1$, where $S_0 \in \mathfrak{S}$ and S_1 is of type kA_1. Note that $\Psi_a(S) \in \mathfrak{S}_2$ if $S \in \mathfrak{S}_2$, for all $a \in S$.

Lemma 4.2. *Let $S \in \mathfrak{S}_2$, $L \in \mathbb{P}(S)$ have type $A_3 + kA_1$, $f \in I(L, S)$. If $f(O) = O$ for a perfect moset O of L, then $f(L) \sim L$.*

Proof. Denote by $L_1 \subset L$ the irreducible component of type A_3. By Proposition 4.2, it suffices to find a $w \in W$ such that $wf(O) = O$ and $wf(O_1) = O_1$ where O_1 is the perfect moset of L_1.

Put
$$Q = L_1 \cup f(L_1), \quad I = Q \cap O, \quad J = O \setminus I.$$

Note that $Q \perp J$. Since Q does not contain any singleton, it belongs to a unique irreducible component $\widetilde{S}$ of $\Psi_J(S)$ (see 3.2.G).

Put $K = O_1 \cap f(O_1)$. If $|K| = 0$, then $|O_1 \cup f(O_1)| = 4$, $|Q| = 5$ or 6 and Q is an acyclic set of type $2A_3$, $\widehat{D_4}$ or $\widehat{D_5}$. In the first case, by Lemma 4.1, there is a $w \in W(\widetilde{S})$ that transposes L_1 and $f(L_1)$. We have $w(O) = O$, $wf(L_1) = L_1$ and $wf(O_1) = O_1$. For the rest of cases O is a moset of a subsystem of type D_4 or D_5, and Theorem 3.3 implies $wf(O_1) = O_1$ for some $w \in W_O$.

If $|K| = 2$, then $f(O_1) = O_1$ and we take $w = \mathrm{Id}$.

Finally, if $|K| = 1$, then $f(O_1) \cap O_1$ is a singleton $\{a\}$. By Theorem 3.2, applied to A_3, we can assume that $f(a) = a$. Let $O_1 = \{a, b\}$. Then $I = \{a, b, c\}$ where $c = f(b)$. We consider $w_1, w_2 \in W(\widetilde{S})_I$ transposing respectively a with b and a with c. Hence $wf(O) = O$ and $wf(O_1) = O_1$, where $w = w_1 w_2$. $\square$

Proof of Proposition 4.4. By induction in $|L|$, we prove a more general statement: if $S \in \mathfrak{S}_2$, then for every irreducible component $L_1 \subset L$ not of type A_1 there is a $w_1 \in W$ such that $w_1 f(L) = L$ and $w_1 f(L_1) = L_1$.

This is trivial for $L = kA_1$. Case $L = A_3 + kA_1$ follows from Lemma 4.2. For the rest of types we consider an irreducible component $L_1 \neq A_1$ and we choose its end e and take $L_1' \subset L_1$ of type A_3 that contains e. Since $L' = L_1' \cup O$ is of type $A_3 + kA_1$, we conclude from Lemma 4.2 that $wf(L') = L'$ for some $w \in W$. Hence $wf(L_1') = L_1'$ and $wf(O) = O$. By Theorem 3.2, applied to A_3, we can choose w such that $wf(e) = e$.

Put $\widetilde{S} = \Psi_e(S)$, $\widetilde{L} = \Psi_e(L)$, $\widetilde{L_1} = \Psi_e(L_1)$. We have $\widetilde{L}, wf(\widetilde{L}) \subset \widetilde{S} \in \mathfrak{S}_2$. If $\widetilde{L_1} \neq A_1$, then by induction, there is a $w_1 \in W(\widetilde{S})$ such that $w_1 wf(\widetilde{L}) = \widetilde{L}$ and $w_1 wf(\widetilde{L_1}) = \widetilde{L_1}$. In particular, $w_1 wf$ preserves $L \setminus L_1$ and the perfect moset of L_1. By Proposition 4.2, $w_2 w_1 wf(L) = L$ and $w_2 w_1 wf(L_1) = L_1$ for a $w_2 \in W$.

Finally, suppose $\widetilde{L_1} = A_1$. Then $L_1 = L_1'$ has type A_3 and $wf(L_1) = L_1$. Set O_1 to be the perfect moset of L_1, $L_2 = L \setminus L_1$ and $S_1 = \Psi_{O_1}(S) \in \mathfrak{S}_2$. Since wf preserves the perfect moset of L_2, we obtain by induction that $w_1 wf(L_2) = L_2$ for a $w_1 \in W(S_1)$. Since $w_1 wf(O_1) = O_1$, Proposition 4.2 implies the existence of a $w_2 \in W$ such that $w_2 w_1 wf(L) = L$ and $w_2 w_1 wf(L_1) = L_1$. $\qquad\square$

4.4 Description of Weyl orbits

To classify Weyl orbits in $\mathbb{P}(S)$ we consider subdiagrams L of $\Delta(S)$ representing elements of $\mathbb{P}(S)$ and we investigate which of them are normal and which of isomorphic special Π-diagrams are conjugate. The general case can be easily reduced to the case of irreducible S. For $S = A_n$ and E_6 a solution is given by:

Theorem 4.4. *All Π-systems in A_n and E_6 are normal.*

This follows from Theorems 3.1 and 3.2.

4.4.1 Case of D_n

In the case of D_n, we call a Π-diagram[17] $L \subset \Delta(D_n)$ *thin* if it contains no pair $\{i, i'\}$, and we call it *thick* if it is not thin. Thick diagrams of type $2A_1$ are conjugate and the same is true for type A_3. We call them D_2- and D_3-diagrams. The number of D_i-subdiagrams of L is denoted by $\delta_i = \delta_i(L)$. We call (δ_2, δ_3) the *tag of L*. We denote by $\omega(L)$ and we call the *width* of a diagram L the minimal value of k such that L is contained in $\Delta(D_k)$. Thin significant diagrams of width n are called *distinguished*. Such diagrams exist only if n is even.

Theorem 4.5. *Every class of isomorphic distinguished diagrams consists of two W-orbits transformed into each other by any automorphism τ_i of $\Delta(D_n)$ transposing i and i' and not moving the rest of nodes. In all other cases isomorphic Π-diagrams are conjugate if and only if they have the same tag.*

Since subdiagrams of any $L \subset \mathbb{P}(D_n)$ of classes D_2 and D_3 are contained in the significant part A of L, the tags of L and A coincide. Besides, all distinguished diagrams are significant. By Theorem 4.1, it is sufficient to prove Theorem 4.5 for significant L.

4.4.A. Let $N = N_1 \cup N_2$ be the partition of the set $N = \{1, 2, 3, \ldots, 2m - 1\}$ into odd and even numbers. Every distinguished subdiagram of $\Delta(D_{2m})$ corresponds to a pair (I, J), where $I \subset N_1$ and $J \subset N_2$, by the formula

$$L_I^J = \tau_I(N \setminus J),$$

---

[17]By a Π-diagram we mean a Dynkin subdiagram of $\Delta(\Sigma)$.

where $\tau_I = \prod_{i \in I} \tau_i$. Diagrams $L_{I_1}^{J_1}$ and $L_{I_2}^{J_2}$ are isomorphic if $J_1 = J_2$, and they are conjugate if and only if, in addition, $|I_1|$ and $|I_2|$ are both odd or both even.

This follows from Theorem 3.3.

4.4.B. Not distinguished significant Π-diagrams are conjugate if and only if they have the same tag.

Proof. Suppose $L' = w(L)$ where $w \in W$. Then L and L' have the same tag because W preserves the types of connected components of a Π-diagram and it preserves the classes of thin and thick diagrams.

On the other hand, if isomorphic L and L' have the same tag, then it is possible to establish a 1-1 correspondence between their components of each class D_2, D_3 and between the thin connected components of each type A_{2i-1}. We get this way an isomorphism $f : L \to L'$. We can choose f such that the restriction of f to the perfect moset of L coincides with a permutation of columns in the matrix (3.1). This permutation belongs to W by Theorem 3.3. By Theorem 3.1, $f \in W$ and $L \sim f(L) = L'$. $\qquad\square$

4.4.2   Case of E_7

We define the parity of $L \in \mathbb{P}(S)$ as the parity of its perfect moset O. Note that the parity does not depend on the choice of L within one Weyl orbit. The same is true for $|O|$ which we call the *charge* of L.

Theorem 4.6. *In the case of E_7 every special orbit belongs to one of three types*

$$A_5, A_3 + A_1, 3A_1 \tag{4.1}$$

with charge 3 or to one of types

$$A_5 + A_1, A_3 + 2A_1, 4A_1 \tag{4.2}$$

with charge 4. Each of these types consists of two Weyl orbits: one with parity 0 and the other with parity 1.

Remark 4.1. Both orbits of charge 3 can be represented by distinguished subdiagrams of any D_6-subdiagram of $\Delta(E_7)$ (for instance, of $L = \{2, 3, 4, 5, 6, 7\}$). A similar representation, with $D_6 + A_1$ (for instance, $L + \{\ell_4\}$) substituted for L, is possible for orbits of charge 4.

By Theorem 4.2 every special L is significant and therefore it has a unique moset O. By Theorem 4.3 and by Proposition 3.2, Weyl orbits are in a 1-1 correspondence with the core orbits. We conclude from Corollary 3.1 that all special Π-systems L have charge 3 or 4 and that the class of all isomorphic special L consists of two W-orbits: one with parity 0, the other with parity 1. All significant types with charge 3 are $3A_1$, $A_3 + A_1$, A_5.

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All significant types with charge 4 are $4A_1$, $A_3 + 2A_1$, $A_5 + A_1$, $2A_3$, A_7. But the latter two types are normal. Indeed, in these cases the group of transformations of O induced by W_O contains a subgroup isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$. Then by Corollary 3.2 the parity of L must be 0 and therefore L is normal. We arrive to the list of six types presented in Table 7.

Table 7: Special Weyl orbits for E_7

Type	Parity 0	Parity 1
$3A_1$	$\{2\} + \{5\} + \{7\}$	$\{3\} + \{5\} + \{7\}$
$A_3 + A_1$	$\{5, 6, 7\} + \{2\}$	$\{5, 6, 7\} + \{3\}$
A_5	$\{2, 4, 5, 6, 7\}$	$\{3, 4, 5, 6, 7\}$
$4A_1$	$\{3\} + \{5\} + \{7\} + \{\ell_4\}$	$\{2\} + \{5\} + \{7\} + \{\ell_4\}$
$A_3 + 2A_1$	$\{5, 6, 7\} + \{3\} + \{\ell_4\}$	$\{5, 6, 7\} + \{2\} + \{\ell_4\}$
$A_5 + A_1$	$\{3, 4, 5, 6, 7\} + \{\ell_4\}$	$\{2, 4, 5, 6, 7\} + \{\ell_4\}$

4.4.3   Case of E_8

Theorem 4.7. *All special $L \in \mathbb{P}(E_8)$ belong to one of five types*

$$A_7, A_5 + A_1, 2A_3, A_3 + 2A_1, 4A_1 \tag{4.3}$$

with charge 4. To each of these types there correspond two Weyl orbits: one with parity 0 and the other with parity 1.[18]

Theorems 4.2, 4.3 and Corollary 3.3 imply that all special L are significant and every special type is represented by two Weyl orbits with charge 4 and different parities.

Every significant L with charge 4 belongs to one of types $4A_1$, $A_3 + 2A_1$, $2A_3$, $A_5 + A_1$ and A_7. These types are special which follows from Table 8.

Table 8: Special Weyl orbits for E_8

Type	Parity 0	Parity 1
$4A_1$	$\{2\} + \{5\} + \{7\} + \{\ell_5\}$	$\{3\} + \{5\} + \{7\} + \{\ell_5\}$
$A_3 + 2A_1$	$\{7, 8, \ell_5\} + \{5\} + \{2\}$	$\{7, 8, \ell_5\} + \{5\} + \{3\}$
$2A_3$	$\{7, 8, \ell_5\} + \{2, 4, 5\}$	$\{7, 8, \ell_5\} + \{3, 4, 5\}$
$A_5 + A_1$	$\{5, 6, 7, 8, \ell_5\} + \{2\}$	$\{5, 6, 7, 8, \ell_5\} + \{3\}$
A_7	$\{2, 4, 5, 6, 7, 8, \ell_5\}$	$\{3, 4, 5, 6, 7, 8, \ell_5\}$

---

[18]Both orbits can be represented by distinguished subdiagrams of any D_8-subdiagram of $\Delta(E_8)$ (for instance, of $L = \{2, 3, 4, 5, 6, 7, 8, \ell_5\}$).

4.5   Order between W-orbits

If Σ is the root system of an algebra $\mathfrak{g}$, then a partial order between conjugacy classes of subalgebras defined in the Introduction can be investigated in terms of an order between W-orbits in $\mathbb{P}(\Sigma)$. Theorems 4.5, 4.6, 4.7 allow to determine if $O_1 \prec O_2$ for any pair of orbits.

Every normal orbit is determined by its type. For special orbits we use additional labels. For instance, we label twelve special orbits in $\mathbb{P}(E_7)$ represented on Table 7 by a combination of their types and parities (indicated by a superscript 0 or 1 at a type symbol). E. g., writing $[A_5 + A_1]^0$ means type $A_5 + A_1$ and parity 0. Partial order between the corresponding orbits [which follows immediately from Table 7] is demonstrated by the following graph where an arrow is directed from label of O_2 to label of O_1 if $O_1 \prec O_2$.

Order between special orbits for E_7

$$
\begin{array}{ccccc}
[A_5 + A_1]^0 & \longrightarrow & [A_3 + 2A_1]^0 & \longrightarrow & [4A_1]^0 \\
\downarrow & & \downarrow & & \downarrow \\
[A_5]^1 & \longrightarrow & [A_3 + A_1]^1 & \longrightarrow & [3A_1]^1 \\
& & \uparrow & & \uparrow \\
[A_5 + A_1]^1 & \longrightarrow & [A_3 + 2A_1]^1 & \longrightarrow & [4A_1]^1 \\
\downarrow & & \downarrow & & \downarrow \\
[A_5]^0 & \longrightarrow & [A_3 + A_1]^0 & \longrightarrow & [3A_1]^0
\end{array}
$$

A similar directed graph for E_8 consists of two isomorphic connected components. Labels in each component are of the same parity.

Order between special orbits for E_8 ($i = 0$ or 1)

$$
\begin{array}{ccccc}
[A_7]^i & \longrightarrow & [2A_3]^i & & \\
\downarrow & & \downarrow & & \\
[A_5 + A_1]^i & \longrightarrow & [A_3 + 2A_1]^i & \longrightarrow & [4A_1]^i
\end{array}
$$

Analogous graphs can be easily drawn for D_n with not too big n.

For any ADE system and normal O_1, O_2, a relation $O_1 \prec O_2$ holds if and only if the Dynkin diagram of O_1 is a subdiagram of the enhanced Dynkin diagram of O_2. This criterion works also in the case of a special O_2 and normal O_1.

It remains to consider the case of special O_1 and normal O_2. We illustrate an approach to this case on the following example. Suppose $\Sigma = E_8$ and labels of O_1 and O_2 are $[4A_1]^0$ and E_6. Subdiagram $\{2, 3, \ell_1, \ell_2\}$ of $\Delta(E_6)$ has parity 0 as a subdiagram of $\Delta(E_8)$. Therefore $O_1 \prec O_2$. On the other hand, O_1' with label $[4A_1]^1$ is not comparable with O_2 because $4A_1$ is normal in E_6 and therefore its parity in E_8 is 0.

A Appendix

A.1 Basic facts

A finite set Σ of nonzero vectors in a Euclidean space V is called a *root system* if:

(1) Σ spans V;

(2) for every $\alpha, \beta \in \Sigma$,
$$(\alpha|\beta) = \frac{2(\alpha, \beta)}{(\beta, \beta)}$$
is an integer;

(3) for every $\alpha, \beta \in \Sigma$,
$$s_\beta(\alpha) = \alpha - (\alpha|\beta)\beta \in \Sigma;$$

(4) for every $\alpha \in \Sigma$, $k\alpha \notin \Sigma$ for all $k \neq \pm 1$.

Condition (4) not always is a part of the definition of root systems (see, e.g.,[Ser01]). What we call a root system, many authors call a reduced root system.

The *Weyl group* W of Σ is a subgroup of $\mathbb{S}(\Sigma)$ generated by all s_α, $\alpha \in \Sigma$. An *automorphism* of $X \subset \Sigma$ is an element of $\mathbb{S}(X)$ preserving the inner product $(\cdot, \cdot)$. We have $W \subset \mathrm{Aut}(\Sigma)$. A *basis* of Σ is a subset $\Pi \subset \Sigma$ such that every $\alpha \in \Sigma$ has a unique presentation $\sum_{\gamma \in \Pi} k_\gamma \gamma$, where all k_γ are non-negative integers or all non-positive. Any root system contains a basis.

Let Σ be a root system and W its Weyl group. We say that $X, Y \subset \Sigma$ are conjugate, if $w(X) = Y$ for some $w \in W$.

Proposition A.1. *All bases of Σ are conjugate (*[Hum90, Theorem 1.4]*).*

Proposition A.2. *If Σ is irreducible, then all roots of Σ having the same length are conjugate (*[Bou02, Chapter VI, Proposition 11]*).*

Proposition A.3. *If $w(\alpha) = \alpha$, where $\alpha \in \Sigma$ and $w \in W$, then there are roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ orthogonal to α, such that $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$ (*[Hum90, Theorem 1.12]*).*

Proposition A.4. *Let $\Pi \subset \Sigma$ be a basis. All automorphisms of Π extend uniquely to automorphisms of Σ, forming a subgroup $A \subset \mathrm{Aut}\,\Sigma$. The group $\mathrm{Aut}(\Sigma)$ is a semidirect product of A and the normal subgroup W (*[Dyn51, Theorem 1]*).*

Proposition A.5. *If Σ is irreducible not of type D_{2n}, $n \geq 2$, then $\mathrm{Aut}\,\Sigma$ is generated by W and $-\mathrm{Id}$ (*[Dyn51, Lemma 5] *or* [Dyn52a, Theorem 0.16]*).*

Proposition A.6. *Suppose Λ is an extended basis of an irreducible Σ. If $\Lambda \setminus \{\alpha\}$ and $\Lambda \setminus \{\beta\}$ are bases, then $w(\Lambda) = \Lambda$ and $w(\alpha) = \beta$ for some $w \in W$.*

This follows from the list of automorphisms for extended Dynkin diagrams and Proposition A.4.

A.2 Notation Index

$\mathrm{Aut}(X)$: group of automorphisms of X

$f|_X$: restriction of a map f to X

G_X: subgroup $\{g \in G : g(X) = X\}$ of a group G

Id: identity map

$I(X, \Sigma)$: set of all embeddings of X into Σ

$p(\alpha)$: projective root $(\alpha, -\alpha)$ corresponding to a root α

$\mathbb{P}$: set of all Π-systems (in Σ or in S)

S : projective root system

$\mathbb{S}(X)$: group of all permutations on a finite set X

W : Weyl group

$W(X, \Sigma)$: set of all $f \in I(X, \Sigma)$ such that $f|_X = w|_X$ for some $w \in W$

Γ_X: diagram of X

Δ_X: projective diagram of X

$\Delta(\Sigma)$: enhanced Dynkin diagram of Σ

$\Theta_X(\Sigma)$: irreducible component of $\Psi_X(\Sigma)$ not of type A_1; $\emptyset$ if there is no such a component

Σ : root system

$\Psi_X(\Lambda)$: orthogonal complement of X in Λ

$\widehat{\Lambda}$: extended Π-system for an irreducible Π-system Λ

$\overline{X}$: completion of X

$|X|$: cardinality of X

$X \simeq Y$: isomorphic X and Y

$X \sim Y$: conjugate X and Y

(α, β): entries $\frac{2(\alpha,\beta)}{(\beta,\beta)}$ of Cartan matrix

$\langle a|b \rangle$: $|(\alpha|\beta)|$ for every $a = (\alpha, -\alpha), b = (\beta, -\beta)$

$\perp$: orthogonal (with respect to $(\cdot|\cdot)$ or to $\langle\cdot|\cdot\rangle$)

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